The contrivances of shuffle products and their siblings

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Gérard H. E. Duchamp[*], Jean-Yves Enjalbert[*]
Vincel Hoang Ngoc Minh[*‡], Christophe Tollu[*]

[*] LIPN, Institut Galilée - UMR CNRS 7030,
93430 Villetaneuse, France.

[‡] University Lille 2, 1 Place Déliot, 59024 Lille, France.

1 Introduction

Let X be an alphabet, R an unitary (commutative) ring and let X^* be the monoid freely generated by X.
As a matter of fact, mathematics (in particular in number theory), physics and other sciences provide, for their theories, algebras of functions indexed by words with a product following a simple recursion of the type

$$au \sqcup\!\sqcup_\phi vb = a(u \sqcup\!\sqcup_\phi bv) + b(au \sqcup\!\sqcup_\phi v) + \phi(a,b)(u \sqcup\!\sqcup_\phi v)\ . \tag{1}$$

We will here use a gradation in the complexity (see appendix A)

i) Type I : factor ϕ comes from a product (possibly with zero) between letters (i.e. $X \cup \{0\}$ is a semigroup)

ii) Type II : factor ϕ comes from the deformation of a semigroup product by a bicharacter

iii) Type III : factor ϕ comes from the deformation of a semigroup product by a colour factor

iv) Type IV : factor ϕ is a commutative a law of associative algebra (CAA) on $R.X$

v) Type V : factor ϕ is a law of associative algebra (AA) on $R.X$

Many shuffle products arise in number theory when studying polylogarithms, harmonic sums and polyzêtas : to study all these products, two of us introduced the Type IV (see above) [11].
On the other hand, in combinatorial physics, one has coproducts with bi-multplicative perturbation factors see [9]).
In a first part, we enumerate some of this product we can meet in the aboves case, to prove the pertinence to study this classe of product. The first steep to use a

2 a small zoology of ϕ-shuffle product

2.1 the definition

Let us so give below some examples.

Example 1 (see [17])**.** Product of interated integrals.
As remarked by Chen [4] in order to implement his theory, the product of two iterated integrals can be computed (thanks to the formula of integration by parts) by "shuffling" the differential forms on which it is based. This shuffle reduces on the indices to a bilinear product such that :

$$\begin{aligned} &\forall w \in X^*, && w \sqcup\!\sqcup 1_{X^*} = 1_{X^*} \sqcup\!\sqcup w = w, \\ \textit{and}\quad &\forall a,b \in X^2, \forall\, u,v \in {X^*}^2, && au \sqcup\!\sqcup vb = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v). \end{aligned}$$

where X is a alphabet and X^* its set of words.
For example, for any letter x_0, x and x' in X,

$$x_0x' \shuffle x_0^2x = x_0x'x_0^2x + 2x_0^2x'x_0x + 3x_0^3x'x + 3x_0^3xx' + x_0^2xx_0x'.$$

Example 2 (see [13])**.** Product of quasi-symmetric functions ($Y = \{y_i\}_{i\in\mathbb{N}_+}$).
In the same manner as in the preceding example, quasi-symmetric functions rule the product of (strict) harmonic sums (their product reduces to the shuffle of their indices with an extra term). The stuffle is a bilinear product on $k\langle Y\rangle$ such that :

$$\begin{aligned} &\forall w \in Y^*, && w \stuffle 1_{Y^*} = 1_{Y^*} \stuffle w = w,\\ \text{and}\quad &\forall y_i, y_j \in X, \forall u, v \in X^*, && y_iu \stuffle y_jv = y_i(u \stuffle y_jv) + y_j(y_iu \stuffle v) + y_{i+j}(u \stuffle v). \end{aligned}$$

In particular,

$$(y_3y_1) \stuffle y_2 \quad = \quad y_3y_1y_2 + y_3y_2y_1 + y_3y_3 + y_2y_3y_1 + y_5y_1.$$

Example 3 ([5])**.** Product of large multiple harmonic sums ($X = \{x_i\}_{i\in\mathbb{N}_+}$).
For large harmonic sums, the product is ruled by the minus-stuffle. It is a bilinear product on $k\langle X\rangle$ such that :

$$\begin{aligned} &\forall w \in X^*, && w \minusstuffle 1_{X^*} = 1_{X^*} \minusstuffle w = w, \text{ and}\\ &\forall x_i, x_j \in X, \forall u, v \in X^*, && x_iu \minusstuffle x_jv = x_i(u \minusstuffle x_jv) + x_j(x_iu \minusstuffle v) - x_{i+j}(u \minusstuffle v). \end{aligned}$$

In particular,

$$(x_3x_1) \minusstuffle x_2 \quad = \quad x_3x_1x_2 + x_3x_2x_1 - x_3x_3 + x_2y_3x_1 - x_5x_1.$$

The associativity of the preceding product is no longer provided by a bicharacter but by a color factor, see Prop. (??).

Example 4 ([2])**.** ($X = \{x_i\}_{i\in\mathbb{N}_+}$).
More generally, following Hoffmann, one can introduce a parameter $q \in k$ and define on $k\langle X\rangle$ the deformed shuffle product by :

$$\begin{aligned} &\forall w \in X^*, && w \stuffle_q 1_{X^*} = 1_{X^*} \stuffle_q w = w,\\ \text{and}\quad &\forall x_i, x_j \in X, \forall u, v \in X^*, && x_iu \stuffle_q x_jv = x_i(u \stuffle_q x_jv) + x_j(x_iu \stuffle_q v) + qx_{i+j}(u \stuffle_q v). \end{aligned}$$

Of course, when $q = -1, 0$ or 1, one obtains respectively the products $\minusstuffle$, $\shuffle$ and $\stuffle$. In particular,

$$(x_3x_1) \stuffle_q x_2 \quad = \quad x_3x_1x_2 + x_3x_2x_1 + q.x_3x_3 + x_2x_3x_1 + qx_5x_1.$$

Example 5 ([10])**.** Product of coloured sums ($X = \{x_i\}_{i\in\mathbb{C}^*}$).
The smuffle is a bilinear product on $k\langle X\rangle$ such that :

$$\begin{aligned} &\forall w \in X^*, && w \smuffle 1_{X^*} = 1_{X^*} \smuffle w = w,\\ \text{and}\quad &\forall x_i, x_j \in X, \forall u, v \in X^*, && x_iu \smuffle x_jv = x_i(u \smuffle x_jv) + x_j(x_iu \smuffle v) + x_{i\times j}(u \smuffle v). \end{aligned}$$

For example,

$$x_{\frac{2}{3}}x_{-1} \smuffle x_{\frac{1}{2}} = x_{\frac{2}{3}}x_{-1}x_{\frac{1}{2}} + x_{\frac{2}{3}}x_{\frac{1}{2}}x_{-1} + x_{\frac{2}{3}}x_{\frac{-1}{2}} + x_{\frac{1}{2}}x_{\frac{2}{3}}x_{-1} + x_{\frac{1}{3}}x_{-1}.$$

Other products can be constructed from these examples. Indeed,

Example 6 ([11])**.** Product of colored polyzêtas ($Y = \{y_i\}_{i\in\mathbb{N}^*}$, $X = \{x_i\}_{i\in\mathbb{C}^*}$).
Let Y and X be two alphabets and consider the alphabet $M = Y \times X$ with the concatenation defined recursively by $(y, x).(w_Y, w_X) = (yw_Y, xw_X)$ for any letters $y \in Y$, $x \in X$, and any words $w_Y \in Y^*$, $w_X \in X^*$ such that $|w_Y| = |w_X|$. The unit of this monoid $\langle M\rangle$ is given by $1_{\langle M\rangle} = (1_{Y^*}, 1_{X^*})$. From the examples 2 and 5, the duffle is defined as a bilinear product n $k\langle M\rangle$ such that

$$\begin{aligned}
\forall w \in M^*, \quad & w \uplus 1_{M^*} = 1_{M^*} \uplus w = w,\\
\forall y_i, y_j \in Y^2, \forall x_k, x_l \in X^2, \forall u, v \in M^{*2}, \quad & (y_i, x_k).u \uplus (y_j, x_l).v = (y_i, x_k)(u \uplus (y_j, x_l)v)\\
& +(y_j, x_l)((y_i, x_k)u \uplus v) + (y_{i+j}, x_{k\times l})(u \uplus v).
\end{aligned}$$

OR better : Product of colored polyzêtas ($X = \{x_{i,k}\}_{(i,k)\in\mathbb{N}^*\times\mathbb{C}^*}$).
Let $(B, +)$ and $(C, \times)$ be two monoids and consider the alphabet $X = \{x_{i,k}\}_{(i,k)\in B\times C}$. the duffle is defined on $k\langle X\rangle$ as a bilinear product such that

$$\begin{aligned}
\forall w \in X^*, \quad & w \uplus 1_{X^*} = 1_{X^*} \uplus w = w,\\
\forall x_{i,k}, x_{j,l} \in X^2, \forall u, v \in X^{*2}, \quad & x_{i,k}.u \uplus x_{j,l}.v = x_{i,k}(u \uplus x_{j,l}v) + x_{j,l}(x_{i,k}u \uplus v) + x_{i+j,k\times l}(u \uplus v).
\end{aligned}$$

Example 7 ([12])**.** Product of Hurwitz polyzêtas ($Y = \{y_i\}_{i\in\mathbb{N}^*}$, $Z = \{z_t\}_{t\in\mathbb{C}}$).
One constructs in the same way the alphabet $N = Y \times Z$ and define on k the product $\circ$ by

$$\begin{aligned}
&\forall w \in N^*, \quad w \circ 1_{N^*} = 1_{N^*} \circ w = w,\\
&\forall y_i, y_j \in Y^2, \forall z_t, z_{t'} \in Z^2, \forall u, v \in N^{*2},\\
&t = t' \Rightarrow (y_i, z_t)u \circ (y_j, z_t)v\\
&\qquad = (y_i, z_t)(u \circ (y_j, z_t)v) + (y_j, z_t)((y_i, z_t)u \circ v)\\
&\qquad +(y_{i+j}, z_t)(u \circ v)\\
&t \neq t' \Rightarrow (y_i, z_t).u \circ (y_j, z_{t'}).v\\
&\qquad = (y_i, z_t).\,(u \circ (y_j, z_{t'}).v) + (y_j, z_{t'}).\,((y_i, z_t).u \circ v)\\
&\qquad + \sum_{n=0}^{i-1} \binom{j-1+n}{j-1} \frac{(-1)^n}{(t-t')^{j+n}} (y_{i-n}, z_t).\,(u \circ v)\\
&\qquad + \sum_{n=0}^{j-1} \binom{i-1+n}{i-1} \frac{(-1)^n}{(t'-t)^{i+n}} (y_{j-n}, z_{t'}).\,(u \circ v)
\end{aligned}$$

*OR better :*Product of Hurwitz polyzêtas ($Y = \{y(i, t)\}_{(i,t)\in\mathbb{N}^*\times\mathbb{C}}$)
Let $(B, +)$ a monoid and C a set and consider the alphabet $Y = \{y_{i,t}\}_{(i,t)\in B\times C}$. The duffle is defined as a bilinear product on $k\langle X\rangle$ such that

$$\begin{aligned}
&\forall w \in Y^*, w \uplus 1_{Y^*} = 1_{Y^*} \uplus w = w,\\
&\forall y_{i,t}, y_{j,t'} \in Y^2, \forall u, v \in Y^{*2}\\
&t = t' \Rightarrow y_{i,t}u \circ y_{j,t}v = y_{i,t}(u \circ y_{j,t}v) + y_{j,t}(y_{i,t}u \circ v) + y_{i+j,t}(u \circ v)\\
&t \neq t' \Rightarrow y_{i,t}.u \circ y_{j,t'}.v\\
&\qquad = y_{i,t}(u \circ y_{j,t'}v) + y_{j,t'}(y_{i,t}u \circ v)\\
&\qquad + \sum_{n=0}^{i-1} \binom{j-1+n}{j-1} \frac{(-1)^n}{(t-t')^{j+n}} y_{i-n,t}.\,(u \circ v)\\
&\qquad + \sum_{n=0}^{j-1} \binom{i-1+n}{i-1} \frac{(-1)^n}{(t'-t)^{i+n}} y_{j-n,t'}.\,(u \circ v)
\end{aligned}$$

Example 8. Product of Generalized Lerch function ($Y = \{y_i\}_{i\in\mathbb{N}^*}$, $X = \{x_i\}_{i\in\mathbb{C}^*}$, $Z = \{z_t\}_{t\in\mathbb{C}}$).
Let X, Y, Z be three alphabets and consider the alphabet $A = Y \times Z \times X$ with the concatenation defined recursively by $(y,z,x).(w_Y,w_Z,w_X) = (yw_Y, zw_Z, xw_X)$ for any letters $y \in Y$, $z \in Z$, $x \in X$, and any words $w_Y \in Y^*$, $w_Z \in Z^*$, $w_X \in X^*$ such that $|w_Y| = |w_Z| = |w_X|$. The unit of this monoid $\langle A\rangle$ is given by $1_{\langle A\rangle} = (1_{Y^*}, 1_{Z^*}, 1_{X^*})$. The product $\overline{\circ}$ is defined by :

$$\begin{array}{ll}
\forall w \in A^*, & w\overline{\circ}1_{A^*} = 1_{A^*}\overline{\circ}w = w,\\
\multicolumn{2}{l}{\forall (y_i,y_j)\in Y^2, \forall(z_t,z_{t'})\in Z^2, \forall(x_k,x_l)\in X^2, \forall(u,v)\in A^{*2},}\\
t = t' \;\Rightarrow & (y_i,z_t,x_k).u\overline{\circ}(y_j,z_t,x_l).v\\
& = (y_i,z_t,x_k).\left(u\overline{\circ}(y_j,z_t).v\right) + (y_j,z_t,x_l).\left((y_i,z_t).u\overline{\circ}v\right)\\
& +(y_{i+j},z_t,x_{k\times l}).\left(u\overline{\circ}v\right)\\
t \neq t' \;\Rightarrow & (y_i,z_t,x_k).u\overline{\circ}(y_j,z_{t'},x_l).v\\
& = (y_i,z_t,x_k).\left(u\overline{\circ}(y_j,z_{t'}).v\right) + (y_j,z_{t'},x_l).\left((y_i,z_t).u\overline{\circ}v\right)\\
& +\displaystyle\sum_{n=0}^{i-1}\binom{j-1+n}{j-1}\frac{(-1)^n}{(t-t')^{j+n}}\,(y_{i-n},z_t,x_{k\times l}).\left(u\overline{\circ}v\right)\\
& +\displaystyle\sum_{n=0}^{j-1}\binom{i-1+n}{i-1}\frac{(-1)^n}{(t'-t)^{i+n}}\,(y_{j-n},z_{t'},x_{k\times l}).\left(u\overline{\circ}v\right)
\end{array}$$

*OR better :*Product of Generalized Lerch function ($X = \{x_{i,k,t}\}_{(i,k,t)\in\mathbb{N}^*\times\mathbb{C}^*\times\mathbb{C}}$)
Let $(B,+)$ a monoid and C a set and consider the alphabet $Y = \{y_{i,t}\}_{(i,t)\in B\times C}$. The duffle is defined as a bilinear product on $k\langle X\rangle$ such that

$$\begin{array}{ll}
\multicolumn{2}{l}{\forall w\in Y^*, w \,⧢\, 1_{Y^*} = 1_{Y^*} \,⧢\, w = w,}\\
\multicolumn{2}{l}{\forall y_{i,t}, y_{j,t'} \in Y^2, \forall u,v \in Y^{*2}}\\
t = t' \;\Rightarrow & y_{i,t}u\circ y_{j,t}v = y_{i,t}(u\circ y_{j,t}v) + y_{j,t}(y_{i,t}u\circ v) + y_{i+j,t}(u\circ v)\\
t \neq t' \;\Rightarrow & y_{i,t}.u\circ y_{j,t'}.v\\
& = y_{i,t}(u\circ y_{j,t'}v) + y_{j,t'}(y_{i,t}u\circ v)\\
& +\displaystyle\sum_{n=0}^{i-1}\binom{j-1+n}{j-1}\frac{(-1)^n}{(t-t')^{j+n}}\,y_{i-n,t}.\left(u\circ v\right)\\
& +\displaystyle\sum_{n=0}^{j-1}\binom{i-1+n}{i-1}\frac{(-1)^n}{(t'-t)^{i+n}}\,y_{j-n,t'}.\left(u\circ v\right)
\end{array}$$

Example 9. *The q-shuffle product is the bilinear operation $⧢_q$ on $\mathbb{N}[q]\langle A\rangle$ recursively defined by*

$$\begin{array}{rcl}
1_{A^*}⧢_q u &=& u⧢_q 1_{A^*} \;=\; u,\\
(au)⧢_q(bv) &=& a(u⧢_q bv) + q^{|au|}b(au⧢_q v),
\end{array}$$

where u,v (resp. a,b) are words (resp. letters) of A^* (resp. A).

In order to grasp more generality in this work, we start from class V which is the most general. The aim of the paper is to give a structure theorem and necessary and sufficient conditions for it. Class V emerges from definition (10) below. Our framework will use a unitary ring as ground set of scalars (and not a field as it would be expected in combinatorics) because the applications require to work with rings of (analytic or arithmetic) functions.

Definition 10. *Let A be a unitary commutative ring, X be an alphabet and $\phi : X \times X \to A\langle X\rangle$ is an arbitrary mapping. Then it exist a unique mapping $\sqcup\!\sqcup_\phi : X^* \times X^* \to A\langle X\rangle$ satisfying the conditions :*

$$(R)\begin{cases} \text{for any} \quad w \in X^*,\ 1_{X^*}\sqcup\!\sqcup_\phi w = w \star 1_{X^*} = w, \\ \text{for any} \quad a, b \in X \text{ and } u, v \in X^*, \\ \qquad au\sqcup\!\sqcup_\phi bv = a(u \sqcup\!\sqcup_\phi bv) + b(au\sqcup\!\sqcup_\phi v) + \phi(a,b)(u\sqcup\!\sqcup_\phi v). \end{cases}$$

With the notations as in Def. (10), we have

Proposition 11. *The recursion (R) defines a unique mapping $X^* \times X^* \to A\langle X\rangle$.*

Definition 12. *We will noted $|l|$ the length of the word l.*

Proof. Make a recurrence over $n = |u| + |v|$. □

From now on, we suppose that ϕ takes its values in AX the space of homogeneous polynomials of degree 1. We still denote by ϕ its linear extension to $AX \otimes AX$ given by

$$\phi(P, Q) = \sum_{x,y\in X} \langle P|x\rangle\langle Q|y\rangle\phi(x, y) \tag{2}$$

and $\star$ the extension of the mapping of Prop (11) by linearity[1] to $A\langle X\rangle \otimes A\langle X\rangle$. Then $\star$ becomes a law of algebra (with 1_{X^*} as unit) on $A\langle X\rangle$.

2.2 extented quasi-stuffle relations

Lemma 13. *For s, r integers, a, b in $\mathbb{C}$:*

$$\forall x \in \mathbb{C} \setminus \{a, b\}, \frac{1}{(x-a)^s(x-b)^r} = \sum_{k=1}^{s} \frac{a_k}{(x-a)^k} + \sum_{k=1}^{t} \frac{b_k}{(x-b)^k}$$

where $a_k = \dfrac{(-1)^{s-k}}{(a-b)^{s+r-k}} \times \dfrac{(s+r-k-1)!}{(r-1)!(n-k)!} = \dbinom{s+r-k-1}{r-1} \dfrac{(-1)^{s-k}}{(a-b)^{s+r-k}}$ *and*
$b_k = \dfrac{(-1)^{r-k}}{(b-a)^{s+r-k}} \times \dfrac{(s+r-k-1)!}{(s-1)!(r-k)!} = \dbinom{s+r-k-1}{s-1} \dfrac{(-1)^{r-k}}{(b-a)^{s+r-k}}$

Let $\mathbf{t} = (t_1, \ldots, t_n)$ a set of parameters, $\mathbf{s} = (s_1, \ldots, s_r)$ a composition, $\phi \in \mathbb{C}^r$, we define, for $n \in \mathbb{Z}_{>0}$,

$$M^n_{\mathbf{s},\phi,\mathbf{t}} \quad = \quad \sum_{n>n_1>\ldots>n_r>0} \prod_{i=1}^{r} \frac{\xi_i^{n_i}}{(n_i - t_i)^{s_i}}. \tag{3}$$

and $M^n_{(),(),()} = 1$.

[1] We recall that AX (resp. $A\langle X\rangle$) admits X (resp. X^*) as linear basis, therefore $AX \otimes AX$ (resp. $A\langle X\rangle \otimes A\langle X\rangle$) is free with basis $X \times X$ (resp. $X^* \times X^*$) or more precisely, the image family $(x \otimes y)_{x,y\in X}$ (resp. $(u \otimes v)_{u,v\in X^*}$).

Proposition 14. *Let* $\mathbf{s} = (s_1, \ldots, s_l)$ *and* $\mathbf{r} = (r_1, \ldots, r_k)$ *two compositions,* $\phi \in \mathbb{C}^l$, $\rho \in \mathbb{C}^k$, *and* $\mathbf{t} = (t_1, \ldots, t_n)$, $\mathbf{t}' = (t'_1, \ldots, t'_n)$ *two sets of parameters. Then*

$$\forall n \in \mathbb{N}, \quad M^n_{\mathbf{s},\phi,\mathbf{t}}\, M^n_{\mathbf{r},\rho,\mathbf{t}'} = M^n_{(\mathbf{s},\phi,\mathbf{t})\overline{\circ}(\mathbf{r},\rho,\mathbf{t}')}.$$

Proof. Put $\mathbf{s}' = (s_2, \ldots, s_l)$, $\mathbf{r}' = (r_2, \ldots, r_k)$, $\phi' = (\xi_2, \ldots, \xi_k)$, $\rho' = (\rho_2, \ldots, \rho_l)$, $\mathbf{t}_2 = (t_2, \ldots, t_n)$ and $\mathbf{t'}_2 = (t'_2, \ldots, t'_n)$ then

$$\begin{aligned}
M^n_{\mathbf{s},\phi,\mathbf{t}}\, M^n_{\mathbf{r},\rho,\mathbf{t}'} &= \sum_{n>n_1, n>n'_1} \frac{\xi_1^{n_1}}{(n_1-t_1)^{s_1}} M^{n_1}_{\mathbf{s}',\phi',\mathbf{t}_2} \frac{\rho_1^{n'_1}}{(n'_1-t'_1)^{r_1}} M^{n'_1-t'_1}_{\mathbf{r}',\rho',\mathbf{t}'_2} \\
&= \sum_{n>n_1} \frac{\xi_1^{n_1}}{(n_1-t_1)^{s_1}} M^{n_1}_{\mathbf{s}',\phi',\mathbf{t}_2} M^{n_1}_{\mathbf{r}',\rho',\mathbf{t}'_2} + \sum_{n>n'_1} \frac{\rho_1^{n'_1}}{(n'_1-t'_1)^{r_1}} M^{n'_1}_{\mathbf{s}',\phi',\mathbf{t}_2} M^{n'_1}_{\mathbf{r}',\rho',\mathbf{t}'_2} \\
&\quad + \sum_{n>m} (\xi_1\rho_1)^m \frac{1}{(n_1-t_1)^{s_1}} \frac{1}{(n'_1-t'_1)^{r_1}} M^m_{\mathbf{s}',\phi'} M^m_{\mathbf{r}',\rho'} \\
&= \sum_{n>n_1} \frac{\xi_1^{n_1}}{(n_1-t_1)^{s_1}} M^{n_1}_{\mathbf{s}',\phi',\mathbf{t}_2} M^{n_1}_{\mathbf{r}',\rho',\mathbf{t}'_2} + \sum_{n>n'_1} \frac{\rho_1^{n'_1}}{(n'_1-t'_1)^{r_1}} M^{n'_1}_{\mathbf{s}',\phi',\mathbf{t}_2} M^{n'_1}_{\mathbf{r}',\rho',\mathbf{t}'_2} \\
&\quad + \sum_{n>m} (\xi_1\rho_1)^m \sum_{k=1}^{t} \frac{1}{(x-b)^k} + \sum_{k=1}^{s} \frac{1}{(x-a)^k} M^m_{\mathbf{s}',\phi'} M^m_{\mathbf{r}',\rho'}.
\end{aligned}$$

A recurrence ended the demonstration. □

Theorem 15. *Let* $\mathbf{s} = (s_1, \ldots, s_l)$ *and* $\mathbf{r} = (r_1, \ldots, r_k)$ *two compositions,* ϕ *a* l*-tuple and* ρ *a* k*-tuple of* $\mathcal{E}$, $\mathbf{t} = (t_1, \ldots, t_k)$ *and* $\mathbf{t}' = (t_1, \ldots, t_k)$ *two* k*-tuple. Then*

(i) For the colored zeta function :

$$\zeta(\mathbf{s}, \xi)\zeta(\mathbf{s}', \xi') \;=\; \zeta\left((\mathbf{s}, \xi) \uplus\!\!\!\sqcup (\mathbf{s}', \xi')\right)$$

(ii) For the Hurwitz zeta function :

$$\zeta(\mathbf{s}, \mathbf{t})\zeta(\mathbf{s}', \mathbf{t}') \;=\; \zeta\left((\mathbf{s}, \mathbf{t}) \circ (\mathbf{s}', \mathbf{t}')\right)$$

(iii) For the Lerch Generalized Function :

$$\zeta(\mathbf{s}, \mathbf{t}, \xi)\zeta(\mathbf{s}', \mathbf{t}', \xi') \;=\; \zeta\left((\mathbf{s}, \mathbf{t}, \xi)\overline{\circ}(\mathbf{s}', \mathbf{t}', \xi')\right)$$

Proof. With $\lambda_n = 1/(n-t)$.

$$\forall n \in \mathbb{N}, \quad M^n_{\mathbf{s},\phi}(\lambda) = \sum_{n>n_1>\ldots>n_r} \prod_{i=1}^{r} \frac{\xi_i^{n_i}}{(n_i - t)^{s_i}}.$$

so $\lim_{n\to\infty} M^n_{\mathbf{s},\phi}(\lambda) = \mathrm{Di}(\mathbf{F}_{\phi,\mathbf{t}}; \mathbf{s})$. The move to limit the proposition 14 gives the result. □

Example 16. *The use of examples 2 and 5 give*

$$\begin{aligned}
&\mathrm{Di}(\mathbf{F}_{(\frac{2}{3},-1),\mathbf{t}}; (3,1))\, \mathrm{Di}(\mathbf{F}_{(\frac{1}{2}),(t)}; (2)) \\
&= \mathrm{Di}(\mathbf{F}_{(\frac{2}{3},-1,\frac{1}{2}),(t,t,t)}; (3,1,2)) + \mathrm{Di}(\mathbf{F}_{(\frac{2}{3},\frac{1}{2},-1),(t,t,t)}; (3,2,1)) \\
&+ \mathrm{Di}(\mathbf{F}_{(\frac{2}{3},-\frac{1}{2}),\mathbf{t}}; (3,3)) + \mathrm{Di}(\mathbf{F}_{(\frac{1}{2},\frac{2}{3},-1),(t,t,t)}; (2,3,1)) \\
&+ \mathrm{Di}(\mathbf{F}_{(\frac{1}{3},-1),\mathbf{t}}; (5,1))
\end{aligned}$$

3 Radford's theorem for the B-shuffle.

Definition 17. *Let I be a set endowed with a length function $l : I \longrightarrow \mathbb{N}$, k be a ring with unit, $\mathcal{A}$ be a k-modulus.*
We will say that a family $(a_i)_{i\in I}$ of $\mathcal{A}$ is triangular compared with the family $(b_i)_{i\in I}$ of $\mathcal{A}$ if

$$\forall i \in I, \exists(\alpha_i^j)_{\{j\in I/l(j)<l(i)\}} \in k^{(\mathbb{N})} / \, a_i = b_i + \sum_{l(j)<l(i)} \alpha_i^j b_j.$$

Note that, with this defition, the family $(\alpha_i^j)_j$ corresponding to a_i is finite even if the set $\{j \in I/l(j) < l(i)\}$ is infinite.

Lemma 18. *Let I be a set endowed with a length function $l : I \longrightarrow \mathbb{N}$, k be a ring with unit, $\mathcal{A}$ be a k-modulus, and $(a_i)_{i\in I}$ a family of elements of $\mathcal{A}$ triangular compared with the family $(b_i)_{i\in I}$ of elements of $\mathcal{A}$. Then,*

(i) If the family $(b_i)_{i\in I}$ is free, so is $(a_i)_{i\in I}$.

(ii) $\forall p \in \mathbb{N}^*, \operatorname{span}\left(\{b_j\}_{\{j\in I/l(j)<p\}}\right) = \operatorname{span}\left(\{a_j\}_{\{j\in I/l(j)<p\}}\right)$.
Consequently, if $(b_i)_{i\in I}$ is a generating family, so is $(a_i)_{i\in I}$.

Proof —

(i) Let $\sum_{i\in J} \beta_i a_i = 0$ be a null linear combination of $(a_i)_{i\in I}$ – so J is a finit subset of I.
If $J = \emptyset$, then $\forall i \in J, \beta_i = 0$.
Otherwise, we suppose that one β_i, $i \in J$ is not null. So, we can define $N = \max_{i\in J}(l(i)/\beta_i \neq 0)$ (all $l(i) \in \mathbb{N}$ and J is finite and not empty). Moreover, by assumption, we can find, for all $j \in J$, a finite familly $(\alpha_i^j)_j$ of elements in k such that $a_i = b_i + \sum_{l(j)<l(i)} \alpha_i^j b_j$. Then,

$$\begin{aligned}
\sum_{i\in J} \beta_i a_i &= \sum_{\substack{i\in J\\ l(i)=N}} \beta_i a_i + \sum_{\substack{i\in J\\ l(i)<N}} \beta_i a_i \\
&= \sum_{\substack{i\in J\\ l(i)=N}} \beta_i \Big(b_i + \sum_{l(j)<N} \alpha_i^j b_j\Big) + \sum_{\substack{i\in J\\ l(i)<N}} \beta_i \Big(b_i + \sum_{l(j)<l(i)} \alpha_i^j b_j\Big) \\
&= \Big(\sum_{\substack{i\in J\\ l(i)=N}} \beta_i b_i\Big) + \Big(\sum_{i\in J} \sum_{l(j)<N} \beta_i \alpha_i^j b_j\Big).
\end{aligned}$$

This sum is finite, null and $(b_i)_{i\in I}$ is free so $\forall i \in \{k \in J, l(k) = N\}, \beta_i = 0$ but that contradicts the definition of N.

(ii) The familly $(a_i)_{i\in I}$ is triangular compared with the familly $(b_i)_{i\in I}$, so

$$\forall p \in \mathbb{N}^*, \operatorname{span}\left(\{a_j\}_{\{j\in I/l(j)<p\}}\right) \subset \operatorname{span}\left(\{b_j\}_{\{j\in I/l(j)<p\}}\right)$$

We juste have to prove, for all $p \in \mathbb{N}$, the propertie $\mathcal{P}(p)$: $\operatorname{span}\left(\{b_j\}_{\{j\in I/l(j)<p\}}\right) \subset \operatorname{span}\left(\{a_j\}_{\{j\in I/l(j)<p\}}\right)$.

- For all element i in I such that $l(i) = 0$, we can read a_i in the form :

$$a_i = b_i + \sum_{l(j)<0} \alpha_i^j \, b_j = b_i$$

because $\forall j \in I, l(j) \in \mathbb{N}$.
So $\forall i \in I, l(i) < 1 \Rightarrow b_i \in \text{span}\left(\{a_j\}_{\{j \in I/l(j)<1\}}\right)$: $\mathcal{P}(1)$ is true.

- Assume that $\mathcal{P}(p)$ is true for a integer p.
Let $i \in I$ such that $l(i) < p+1$. We can find a finite familly $(\alpha_i^j)_j$ of k such that $a_i = b_i + \sum\limits_{l(j)<l(i)} \alpha_i^j \, b_j$. Then, $b_i = a_i - \sum\limits_{l(j)<l(i)} \alpha_i^j \, b_j$.
But, for $j \in J$ such that $l(j) < l(i)$, $l(j) < p$ so so $b_j \in \text{span}\left(\{a_j\}_{\{j \in I/l(j)<p\}}\right) \subset \text{span}\left(\{a_j\}_{\{j \in I/l(j)<p+1\}}\right)$. It follows that the elements a_i and $-\sum_{l(j)<l(i)} \alpha_i^j \, b_j$ are in $\text{span}\left(\{a_j\}_{\{j \in I/l(j)<p+1\}}\right)$ so b_i too; consequently $\mathcal{P}(p+1)$ is true.

So, $\forall p \in \mathbb{N}^*$, $\text{span}\left(\{a_j\}_{\{j \in I/l(j)<p\}}\right) \subset \text{span}\left(\{b_j\}_{\{j \in I/l(j)<p\}}\right)$, and the result is proved.

□

Counter-example 19. *In $k[x, x^{-1}]$, the family $(b_k)_{k \in \mathbb{Z}}$ define by $b_k = x^k$ is a basis, the family $(a_k)_{k \in \mathbb{Z}}$ define by $a_k = x^k - x^{k-1}$ verify the condition but is not a basis (b_k is the infinite sum of the a_j, $j \leq k$) : the condition $l(I) \subset \mathbb{N}$ is necessary.*

For the end of this section A be a commutative ring (with unit) and $\phi : AX \otimes AX \to AX$ an associative law $\oplus$. In this case, the product $\star$ defined in definition 10 will be noted $\sqcup\!\sqcup_\phi$.
Any total ordering $<$ on the alphabet X being given, let $\mathcal{L}yn(X)$ denote the family of Lyndon words [17] constructed in X^* w.r.t. this ordering. The largest framework in which Radford's theorem is true [16] is when ϕ is commutative (and associative).

Lemma 20. *Let u and v two words in X^*. Then it exists $(C_{u,v}^w)_{|w|<|u|+|v|} \in A^{(\mathbb{N})}$ such that :*

$$u \sqcup\!\sqcup_\phi v = u \sqcup\!\sqcup v + \sum_{|w|<|u|+|v|} C_{u,v}^w w.$$

Proof — We will make a recurrence over the propertie $\mathcal{P}(n)$:

$$\forall (u,v) \in (X^*)^2, |u| + |v| < n+1 \Rightarrow (\exists (C_{u,v}^w)_w \in A^{(\mathbb{N})} / u \sqcup\!\sqcup_\phi v = u \sqcup\!\sqcup v + \sum_{|w|<|u|+|v|} C_{u,v}^w w.$$

- $1_{X^*} \sqcup\!\sqcup_\phi 1_{X^*} = 1_{X^*} = 1_{X^*} \sqcup\!\sqcup 1_{X^*}$,
$\forall a \in X, 1_{X^*} \sqcup\!\sqcup_\phi a = a = 1_{X^*} \sqcup\!\sqcup a$ and $a \sqcup\!\sqcup_\phi 1_{X^*} = a = a \sqcup\!\sqcup 1_{X^*}$: the propertie is true $\mathcal{P}(n)$ for $n = 0$ and $n = 1$.

- Assume $\mathcal{P}(n)$ is true for one integer n and let u and v two words of X^* such that $|u| + |v| = n + 1$.
If $u = 1_{X^*}$ or if $v = 1_{X^*}$, $u \sqcup\!\sqcup_\phi v = u \sqcup\!\sqcup v$.
Otherwise, we can write $u = au'$ and $v = bv'$, with a, b in X and u', v' in X^*. Then,

$u \sqcup\!\sqcup_\phi v = a(u' \sqcup\!\sqcup_\phi v) + b(u \sqcup\!\sqcup_\phi v') + \phi(a,b)(u' \sqcup\!\sqcup_\phi v')$.
By inductive hypothesis, we can read $u' \sqcup\!\sqcup_\phi v = u' \sqcup\!\sqcup v + \sum\limits_{|w|<n} C^w_{u',v} w$, and too $u \sqcup\!\sqcup_\phi v' = u \sqcup\!\sqcup v' + \sum\limits_{|w|<n} C^w_{u,v'} w$ and $u' \sqcup\!\sqcup_\phi v' = u' \sqcup\!\sqcup v' + \sum\limits_{|w|<n} C^w_{u',v'} w$ with $(C^w_{u',v})$, $(C^w_{u,v'})$ and $(C^w_{u',v'})$ in $A^{(\mathbb{N})}$. So,

$$\begin{aligned}
u \sqcup\!\sqcup_\phi v &= a\big(u' \sqcup\!\sqcup v + \sum_{|w|<n} C^w_{u',v} w\big) + b\big(u \sqcup\!\sqcup v' + \sum_{|w|<n} C^w_{u,v'} w\big) \\
&\quad + \phi(a,b)\big(u' \sqcup\!\sqcup v' + \sum_{|w|<n} C^w_{u',v'} w\big) \\
&= a(u' \sqcup\!\sqcup v) + b(u \sqcup\!\sqcup v') + \phi(a,b)(u' \sqcup\!\sqcup v') \\
&\quad + \sum_{|w|<n} (C^w_{u',v} aw + C^w_{u,v'} bw + C^w_{u',v'} \phi(a,b) w)
\end{aligned}$$

But $a(u' \sqcup\!\sqcup v) + b(u \sqcup\!\sqcup v') = u \sqcup\!\sqcup v$; moreover $u' \sqcup\!\sqcup v'$ is a linear combinaison of words of length $|u'| + |v'| = n-2$, so $\phi(a,b)(u' \sqcup\!\sqcup v') + \sum\limits_{|w|<n} (C^w_{u',v} aw + C^w_{u,v'} bw + C^w_{u',v'} \phi(a,b) w)$ is a linear combinaison of words of length at most n : $\mathcal{P}(n+1)$ is true.

□

Lemma 21. *Let n be a integer such that $n > 1$ and $u_1, u_2, \ldots, u_n$ words of X^*. Then it exist $(C^w_{u_1,\ldots,u_n})_{|w|<|u_1|+\ldots+|u_n|} \in A^{(\mathbb{N})}$ such that :*

$$u_1 \sqcup\!\sqcup_\phi u_2 \sqcup\!\sqcup_\phi \ldots \sqcup\!\sqcup_\phi u_n = u_1 \sqcup\!\sqcup u_2 \sqcup\!\sqcup \ldots \sqcup\!\sqcup u_n + \sum_{|w|<|u_1|+\ldots+|u_n|} C^w_{u_1,\ldots,u_n} w.$$

Proof — We will make an induction over n.

- For $n = 2$, it comes from lemma 20.
- Assume that $\mathcal{P}(n)$ is true for a integer n and take $n+1$ words $u_1, \ldots, u_{n+1}$ in X^*. Posons $u = u_1 \sqcup\!\sqcup_\phi u_2 \sqcup\!\sqcup_\phi \ldots \sqcup\!\sqcup_\phi u_n$, we can find :
One hand, by induction hypothesis, $(C^w_{u_1,\ldots,u_n})_{|w|<|u_1|+\ldots+|u_n|} \in A^{(\mathbb{N})}$ such that

$$u = u_1 \sqcup\!\sqcup u_2 \sqcup\!\sqcup \ldots \sqcup\!\sqcup u_n + \sum_{|w|<|u_1|+\ldots+|u_n|} C^w_{u_1,\ldots,u_n} w$$

and other hand, by lemma 20, $(C^w_{u,u_{n+1}})_{|w|<|u|+|u_{n+1}|} \in A^{(\mathbb{N})}$ such that

$$u \sqcup\!\sqcup_\phi u_{n+1} = u \sqcup\!\sqcup u_{n+1} + \sum_{|w|<|u|+|u_{n+1}|} C^w_{u,u_{n+1}} w.$$

Hence

$$\begin{aligned}
&u_1 \sqcup\!\sqcup_\phi \ldots \sqcup\!\sqcup_\phi u_{n+1} \\
&= u \sqcup\!\sqcup_\phi u_{n+1} \\
&= u \sqcup\!\sqcup u_{n+1} + \sum_{|w|<|u|+|u_{n+1}|} C^w_{u,u_{n+1}} w
\end{aligned}$$

$$= \big(u_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup u_n + \sum_{|w|<|u_1|+\ldots+|u_n|} C^w_{u_1,\ldots,u_n} w\big) \sqcup\!\sqcup u_{n+1} + \sum_{|w|<|u|+|u_{n+1}|} C^w_{u,u_{n+1}} w$$
$$= u_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup u_n \sqcup\!\sqcup u_{n+1} + \sum_{|w|<|u_1|+\ldots+|u_n|} C^w_{u_1,\ldots,u_n} w \sqcup\!\sqcup u_{n+1} + \sum_{|w|<|u|+|u_{n+1}|} C^w_{u,u_{n+1}} w$$

But the product $w \sqcup\!\sqcup u_{n+1}$ is a linear combinaison of words of lenght $|w| + |u_{n+1}|$ so $\sum\limits_{|w|<|u_1|+\ldots+|u_n|} C^w_{u_1,\ldots,u_n} w \sqcup\!\sqcup u_{n+1}$ is a linear combinaison of word of lenght strictly smaller than $|u_1| + \ldots + |u_n| + |u_{n+1}|$.
So, $\sum\limits_{|w|<|u_1|+\ldots+|u_n|} C^w_{u_1,\ldots,u_n} w \sqcup\!\sqcup u_{n+1} + \sum\limits_{|w|<|u|+|u_{n+1}|} C^w_{u,u_{n+1}} w$ is a linear combinaison of word of of lenght strictly smaller than $|u_1|+\ldots+|u_n|+|u_{n+1}|$: the propertie $\mathcal{P}(n+1)$ is true.

□

We will note, for $\alpha \in \mathbb{N}^{(\mathcal{L}yn(X))}$, $||\alpha|| = \sum\limits_{l\in\mathcal{L}yn(X)} \alpha(l)|l|$ and, if $\alpha = (\alpha_1, \ldots, \alpha_r)$, with $l_1, l_2, \ldots$ the ordonned Lyndon words, $\mathbb{X}^{\star\alpha}$ will be $l_1^{\star\alpha_1} \star \ldots \star l_r^{\star\alpha_r}$, where $l^{\star n} = \overbrace{l \star \ldots \star l}^{ntimes}$.

Lemma 22. $\forall \alpha \in \mathbb{N}^{(\mathcal{L}yn(X))}, \exists (C^\alpha_\beta)_\beta \in A^{(\mathbb{N}^{(\mathcal{L}yn(X))})} / \mathbb{X}^{\sqcup\!\sqcup_\phi \alpha} = \mathbb{X}^{\sqcup\!\sqcup\, \alpha} + \sum\limits_{\substack{\beta\in\mathbb{N}^{(\mathcal{L}yn(X))}\\ ||\beta||<||\alpha||}} C^\alpha_\beta \mathbb{X}^{\sqcup\!\sqcup\, \beta}$.

Proof — Let $\alpha \in \mathbb{N}^{(\mathcal{L}yn(X))}$. Apply lemma 21 over Lyndon Words counted with their multiplicity, we find :
$$\mathbb{X}^{\sqcup\!\sqcup_\phi \alpha} = \mathbb{X}^{\sqcup\!\sqcup\, \alpha} + \sum_{w<||\alpha||} C^\alpha_w w.$$
But $\mathcal{L}yn(X)$ was a transcendental basis over X, all word w can be writen $\sum\limits_{\beta_w\in\mathbb{N}^{(\mathcal{L}yn(X))}} \mathbb{X}^{\sqcup\!\sqcup\, \beta_w}$; moreover (by $\sqcup\!\sqcup$ propertie) $||\beta_w|| < |w| + 1$ for all β_w in the decomposition. Put its combinaisons in $\sum\limits_{w<||\alpha||} C^\alpha_w w$, we obtain the result. □

Theorem 23. *Let A be a commutative ring (with unit) and $\phi : AX \otimes AX \to AX$ be an associative and commutative law on AX. Then*

i) If $\mathbb{Q} \subset A$ (i.e. all integers are invertible[2] in A), the algebra $\mathcal{A} = (A\langle X\rangle, \sqcup\!\sqcup_\phi, 1_{X^*})$ is a polynomial algebra which admits $Lyn(X)$ as a transcendence basis.

ii) [11] This algebra $(A\langle X\rangle, \sqcup\!\sqcup_\phi, 1_{X^*})$ can moreover be endowed with the comultiplication Δ_{conc} dual to the concatenation

$$\Delta_{\mathrm{conc}}(w) = \sum_{uv=w} u \otimes v \tag{4}$$

[2]Precisely, $\mathbb{N}^+.1_A \subset A^\times$

and the "constant term" character $\epsilon(P) = \langle P|1_{X^*}\rangle$.
With this setting

$$\mathcal{B}_\phi = (A\langle X\rangle, ⧢_\phi, 1_{X^*}, \Delta_{\mathrm{conc}}, \epsilon) \tag{5}$$

is a bialgebra [3].

Remark 24. *i) It is necessary to suppose $\mathbb{Q} \subset A$ as, in case $\phi \equiv 0$, one has*

$$a^n = \frac{1}{n!}(a^{⧢\, n}) \tag{6}$$

ii) The operator (reduced coproduct)

$$\Delta^+_{\mathrm{conc}}(w) = \sum_{\substack{uv=w\\ u,v\neq 1}} u \otimes v$$

being locally nilpotent, the bialgebra (5) is, in fact a Hopf Algebra.

iii) When $⧢_\phi$ is dualizable the graded dual of $\mathcal{B}_\phi$ is a Hopf algebra.

Proof —

i) $\mathbb{B} = (\mathbb{X}^{⧢\,\alpha})_{\alpha\in\mathbb{N}^{(\mathcal{L}yn(X))}}$ is a basis of A-modulus $\mathcal{A}$ and, by lemma 22, the family $(\mathbb{X}^{⧢_\phi \alpha})_{\alpha\in\mathbb{N}^{(\mathcal{L}yn(X))}}$ is triangular compared with the family $\mathbb{B}$. So, by lemma 18, $(\mathbb{X}^{⧢_\phi \alpha})_{\alpha\in\mathbb{N}^{(\mathcal{L}yn(X))}}$ is a basis of A-modulus $\mathcal{A}$, then $Lyn(X)$ is a transcendence basis of $\mathcal{A}$.

ii) it is a classical combinatoric verifycation, do in [11].

□

4 Condition of B-shuffle

Theorem 25. *(i) The law $⧢_\phi$ is commutative if and only if the extension $\phi : AX \otimes AX \to AX$ is so.*

(ii) The law $⧢_\phi$ is associative if and only if the extension $\phi : AX \otimes AX \to AX$ is so.

Proof —

(i) [$⧢_\phi$ commutative $\Longrightarrow$ ϕ commutative]
Suppose that

$$\forall (u,v) \in (X^*)^2, \qquad u ⧢_\phi v = v ⧢_\phi u.$$

[3] Commutative and, when $|X| \geq 2$, noncocommutative.

In particular, $\forall (x,y)\in (X^*)^2, x\sqcup\!\sqcup_\phi y = x\sqcup\!\sqcup_\phi y$. But, for any $(x,y)\in X^2$,

$$x\sqcup\!\sqcup_\phi y = xy + yx + \phi(x,y) \quad \text{and} \quad y\sqcup\!\sqcup_\phi x = yx + xy + \phi(y,x).$$

and so $(\forall x,y\in X)(\phi(x,y)=\phi(y,x))$.

[ϕ commutative $\Longrightarrow$ $\sqcup\!\sqcup_\phi$ commutative] Now suppose ϕ is commutative then let us prove by reccurence on $|uv|$ that $\sqcup\!\sqcup_\phi$ is commutative :

- The previous equivalence prove the recurrence holds for $|u|=|v|=1$.
- Suppose the recursive stands for any any $u,v\in X^*$ such that $2\le |uv|\le n$ and $|u|,|v|\neq 1$.
- Let $u=xu'$ and $v=yv'$ with $x,y\in X$ and $u',v'\in X^*$. Then,

$$\begin{aligned} u\sqcup\!\sqcup_\phi v &= x(u'\sqcup\!\sqcup_\phi yv) + y(xu'\sqcup\!\sqcup_\phi v) + \phi(x,y)(u'\sqcup\!\sqcup_\phi v') \\ &= x(yv\sqcup\!\sqcup_\phi u) + y(v'\sqcup\!\sqcup_\phi xu') + \phi(y,x)(v'\sqcup\!\sqcup_\phi u') \qquad \text{(by induction hypothesis)} \\ &= v\sqcup\!\sqcup_\phi u. \end{aligned}$$

(ii) [$\sqcup\!\sqcup_\phi$ associative $\Longrightarrow$ ϕ associative] Suppose that

$$\forall u,v,w\in X^*, \qquad (u\sqcup\!\sqcup_\phi v)\sqcup\!\sqcup_\phi w = u\sqcup\!\sqcup_\phi (v\sqcup\!\sqcup_\phi w).$$

Then, for any $x,y,z\in X$, one has

$$(x\sqcup\!\sqcup_\phi y)\sqcup\!\sqcup_\phi z = x\sqcup\!\sqcup_\phi (y\sqcup\!\sqcup_\phi z).$$

But

$$\begin{aligned} (x\sqcup\!\sqcup_\phi y)\sqcup\!\sqcup_\phi z &= (xy + yx + \phi(x,y))\sqcup\!\sqcup_\phi z \\ &= xy\sqcup\!\sqcup_\phi z + yx\sqcup\!\sqcup_\phi z + \phi(x,y)\sqcup\!\sqcup_\phi z \\ &= x(y\sqcup\!\sqcup_\phi z) + z(xy\sqcup\!\sqcup_\phi 1) + \phi(x,z)y + y(x\sqcup\!\sqcup_\phi z) + z(yx\sqcup\!\sqcup_\phi 1) + \phi(y,z)x \\ &+ \phi(x,y)z + z\phi(x,y) + \phi(\phi(x,y),z) \\ &= x(yz + zy + \phi(y,z)) + zxy + \phi(x,z)y + y(xz + zx + \phi(x,z)) + zyx \\ &+ \phi(y,z)x + \phi(x,y)z + z\phi(x,y) + \phi(\phi(x,y),z) \end{aligned}$$

$$\begin{aligned} x\sqcup\!\sqcup_\phi (y\sqcup\!\sqcup_\phi z) &= x\sqcup\!\sqcup_\phi (yz + zy + \phi(y,z)) \\ &= x\sqcup\!\sqcup_\phi yz + x\sqcup\!\sqcup_\phi zy + x\sqcup\!\sqcup_\phi \phi(y,z) \\ &= x(1\sqcup\!\sqcup_\phi yz) + y(x\sqcup\!\sqcup_\phi z) + \phi(x,y)z + x(1\sqcup\!\sqcup_\phi zy) + z(x\sqcup\!\sqcup_\phi y) + \phi(x,z)y \\ &= x\phi(y,z) + \phi(y,z)x + \phi(x,\phi(y,z)) \\ &= xyz + y(xz + zx + \phi(x,z)) + \phi(x,y)z + xzy + z(xy + yx + \phi(x,y)) + \phi(x,z)y \\ &+ x\phi(y,z) + \phi(y,z)x + \phi(x,\phi(y,z)). \end{aligned}$$

One can deduce then

$$(\forall x,y,z\in X)(x\sqcup\!\sqcup_\phi (y\sqcup\!\sqcup_\phi z) = (x\sqcup\!\sqcup_\phi y)\sqcup\!\sqcup_\phi z) \quad\Longleftrightarrow\quad (\forall x,y,z\in X)(\phi(x,\phi(y,z)) = \phi(\phi(x,y),z)).$$

[ϕ associative $\Longrightarrow$ $\sqcup\!\sqcup_\phi$ associative] Now suppose ϕ is associative then let us prove by reccurence on $|uvw|$ that $\sqcup\!\sqcup_\phi$ is associative :

- The previous equivalence prove the recurrence holds for $|u|=|v|=|w|=1$.

– Suppose the recursive stands for any $u, v \in X^*$ such that $3 \leq |uvw| \leq n$ and $|u|, |v|, |w| \neq 1$.
Let $u = xu, v = yv'$ and $w = zw'$ with $x, y, z \in X$ and $u', v', w' \in X^*$. Then,

$$\begin{aligned}
&u ⧢_\phi (v ⧢_\phi w)\\
&= u ⧢_\phi \big(y(v' ⧢_\phi w) + z(v ⧢_\phi w') + \phi(y,z)(v' ⧢_\phi w')\big)\\
&= x(u' ⧢_\phi y(v' ⧢_\phi w)) + y(u ⧢_\phi (v' ⧢_\phi w)) + \phi(x,y)(u' ⧢_\phi (v' ⧢_\phi w))\\
&+x(u' ⧢_\phi z(v ⧢_\phi w')) + z(u ⧢_\phi (v ⧢_\phi w')) + \phi(x,z)(u' ⧢_\phi (v ⧢_\phi w'))\\
&+x(u' ⧢_\phi \phi(y,z)(v' ⧢_\phi w')) + \phi(y,z)(u ⧢_\phi (v' ⧢_\phi w')) + \phi(x,\phi(y,z))u' ⧢_\phi (v' ⧢_\phi w')\\
&= x(u' ⧢_\phi (v ⧢_\phi w))\\
&+y(u ⧢_\phi (v' ⧢_\phi w)) + \phi(x,y)(u' ⧢_\phi (v' ⧢_\phi w))\\
&+z(u ⧢_\phi (v ⧢_\phi w')) + \phi(x,z)(u' ⧢_\phi (v ⧢_\phi w'))\\
&+\phi(y,z)(u ⧢_\phi (v' ⧢_\phi w')) + \phi(x,\phi(y,z))u' ⧢_\phi (v' ⧢_\phi w')
\end{aligned}$$

and

$$\begin{aligned}
&(u ⧢_\phi v) ⧢_\phi w)\\
&= (x(u' ⧢_\phi v) + y(u ⧢_\phi v') + \phi(x,y)(u' ⧢_\phi v')) ⧢_\phi w)\\
&= x((u' ⧢_\phi v) ⧢_\phi w) + z(x(u' ⧢_\phi v) ⧢_\phi w') + \phi(x,z)((u' ⧢_\phi v) ⧢_\phi w')\\
&+y((u ⧢_\phi v') ⧢_\phi w) + z(y(u ⧢_\phi v') ⧢_\phi w') + \phi(y,z)((u ⧢_\phi v') ⧢_\phi w')\\
&+\phi(x,y)((u' ⧢_\phi v') ⧢_\phi w) + z(\phi(x,y)(u' ⧢_\phi v') ⧢_\phi w') + \phi(\phi(x,y),z)((u' ⧢_\phi v') ⧢_\phi w')\\
&= x((u' ⧢_\phi v) ⧢_\phi w) + \phi(x,z)((u' ⧢_\phi v) ⧢_\phi w')\\
&+y((u ⧢_\phi v') ⧢_\phi w) + \phi(y,z)((u ⧢_\phi v') ⧢_\phi w')\\
&+\phi(x,y)((u' ⧢_\phi v') ⧢_\phi w) + \phi(\phi(x,y),z)((u' ⧢_\phi v') ⧢_\phi w')\\
&+z(u ⧢_\phi v) ⧢_\phi w')
\end{aligned}$$

indeed, thanks to the induction hypothesis and the fact ϕ associative, its egal.

□

5 Annex A

Name	Formula (recursion)	ϕ	Type
Shuffle	$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v)$	$\phi \equiv 0$	I
Stuffle	$x_i u \sqcup\!\sqcup x_j v = x_i(u \sqcup\!\sqcup x_j v) + x_j(x_u \sqcup\!\sqcup v) + x_{i+j}(u \sqcup\!\sqcup v)$	$\phi(x_i, x_j) = x_{i+j}$	I
Muffle	$x_i u \sqcup\!\sqcup x_j v = x_i(u \sqcup\!\sqcup x_j v) + x_j(x_u \sqcup\!\sqcup v) + x_{i\times j}(u \sqcup\!\sqcup v)$	$\phi(x_i, x_j) = x_{i\times j}$	I
q-stuffle	$x_i u \sqcup\!\sqcup x_j v = x_i(u \sqcup\!\sqcup x_j v) + x_j(x_u \sqcup\!\sqcup v) + q^{i\times j} x_{i+j}(u \sqcup\!\sqcup v)$	$\phi(x_i, x_j) = q^{i\times j} x_{i+j}$	II
Hoffmann	$x_i u \sqcup\!\sqcup x_j v = x_i(u \sqcup\!\sqcup x_j v) + x_j(x_u \sqcup\!\sqcup v) + q x_{i+j}(u \sqcup\!\sqcup v)$	$\phi(x_i, x_j) = q x_{i+j}$	III
Min-stuffle	$x_i u \sqcup\!\sqcup x_j v = x_i(u \sqcup\!\sqcup x_j v) + x_j(x_u \sqcup\!\sqcup v) - x_{i+j}(u \sqcup\!\sqcup v)$	$\phi(x_i, x_j) = -x_{i+j}$	III
B-stuffle	$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v) + \phi(a,b)(u \sqcup\!\sqcup v)$	$\phi(a,b) = \phi(b,a)$	IV
Semigroup- -stuffle	$x_t u \sqcup\!\sqcup x_s v = x_t(u \sqcup\!\sqcup x_s v) + x_s(x_t u \sqcup\!\sqcup v) + x_{t\perp s}(u \sqcup\!\sqcup v)$	$\phi(x_t, x_s) = x_{t\perp s}$	I
$\mathtt{LDIAG}(1, q_s)$ (non-crossed, non-shifted)	$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v) + q_s^{\lvert a\rvert\lvert b\rvert} a.b(u \sqcup\!\sqcup v)$	$\phi(a,b) = q_s^{\lvert a\rvert\lvert b\rvert}(a.b)$	II
ϕ-stuffle	$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v) + \phi(a,b)(u \sqcup\!\sqcup v)$	$\phi(a,b)$ AAU	V

6 Annex B

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